Asymmetric Factorizations of Matrix Functions on the Real Line

L.P. Castro, R. Duduchava, and F.-O. Speck

Departamento de Matemática
Instituto Superior Técnico
Lisboa, Portugal
Preprint 13/2005

http://preprint.math.ist.utl.pt/?serie=dmist

Asymmetric Factorizations of Matrix Functions on the Real Line

L.P. Castro, R. Duduchava and F.-O. Speck

Dedicated to I.B. Simonenko on the occasion of his 70th birthday

Abstract. We indicate a criterion for some classes of continuous matrix functions on the real line with a jump at infinity to admit both, a classical right and an asymmetric factorization. It yields the existence of generalized inverses of matrix Wiener-Hopf plus Hankel operators and provides precise information about the asymptotic behavior of the factors at infinity and of the solutions to the corresponding equations at the origin.

Mathematics Subject Classification (2000). Primary 47A68; Secondary 15A23, 47B35, 15A21.

Keywords. Right factorization, asymmetric factorization, anti-symmetric factorization.

1. Introduction

In 1968, I.B. Simonenko published his celebrated paper *Some general questions in the theory of the Riemann boundary problem* [Si] that gave rise to intensive studies on Riemann problems, singular integral and Toeplitz operators, etc. including the concepts of generalized factorization [ClGo], Φ–factorization [LiSp] and Wiener-Hopf factorization [BöSi]. In that paper, I. Simonenko gave a rather general definition of factorization of matrices with measurably functions as entries. He proved equivalence of generalized factorization with the solvability of the corresponding systems of singular integral operators and gave many properties of generalized factorization. The paper [Si] continuous to influence the investigations more than four decades already.

This article was started during the second author's visit to Instituto Superior Técnico, U.T.L., and Universidade de Aveiro, Portugal, in February–March 2005.

Among the pioneering works on the subject one should mention contributions by T. Carleman, N. Wiener and H. Hopf, F. Gakhov, N. Muskhelishvili, M. Krein, I. Gohberg, I. Simonenko and many others. See also [GoKaSp] for a survey on matrix-valued functions factorization.

During the last decades, different types of matrix factorizations revealed to be a powerful tool for solving explicitly many problems, e.g. in mathematical physics. Recent work on applications in diffraction theory [CaSpTe1, CaSpTe3] initiated a detailed investigation of Wiener–Hopf plus Hankel operators in spaces of Bessel potentials and their theoretical background.

The present paper continues the investigation started in [CaSpTe2, CaSp]. Some related results on factorization of matrix symbols of pseudodifferential operators are exposed in [ChDu, Sh]. Corresponding work for the circle instead of $\mathbb{R}$ and the factorization theory for Toeplitz plus Hankel operators can be found in [Eh]. The present environment is designed for further applications in mathematical physics as started in [CaSpTe1].

Here we devote particular attention to factorization of matrix-valued functions with discontinuity at infinity, which plays a crucial role in solving some problems of mathematical physics. We establish a criterion for such matrix-valued functions on the real line admit, both, an asymmetric and a classical right factorization. It yields the existence of generalized inverses of *matrix convolution type operators with symmetry* [CaSpTe2] (or Wiener-Hopf plus/minus Hankel operators), and provides precise information about the asymptotical behavior of the factors at infinity, and of the solutions to the corresponding equations at the origin.

2. Classical Factorization

Let $\mathscr{A}$ be a bounded matrix-valued function which belongs to the Zygmund space $\mathscr{Z}^{\mu}(\overline{\mathbb{R}})$ or to the algebra $\mathscr{H}_0^{\mu}(\overline{\mathbb{R}})$, $\mu > 0$ (see Appendix, § A.2) and is supposed to be elliptic:

$$\inf_{x\in\mathbb{R}} |\det \mathscr{A}(x)| > 0 \tag{2.1}$$

The limits $\mathscr{A}(+\infty)$ and $\mathscr{A}(-\infty)$ might differ (in contrast to the case $\mathscr{A} \in \mathscr{Z}^{\mu}(\mathbb{R})$ or $\mathscr{A} \in \mathscr{H}_0^{\mu}(\mathbb{R})$ when these limits coincide) and we consider the *Jordan normal decomposition* of the matrix

$$\mathscr{A}_\infty := [\mathscr{A}(+\infty)]^{-1}\mathscr{A}(-\infty) = \mathscr{K}\,\Lambda_{\mathscr{A}_\infty} B_{\mathscr{A}_\infty}(1)\,\mathscr{K}^{-1}\,. \tag{2.2}$$

Here $\Lambda_{\mathscr{A}_\infty}$ is a diagonal matrix of eigenvalues of $\mathscr{A}_\infty$, $B_{\mathscr{A}_\infty}(1)$ is upper triangular with entries 1 on the main diagonal and $\mathscr{K}$ is an elliptic (det $\mathscr{K} \neq 0$) transformation matrix (see Appendix, § A.1 for details).

Let $\lambda_1, \ldots, \lambda_\ell$ be all eigenvalues of the matrix $\mathscr{A}_\infty$ with the Riesz indices $m_1, \cdots,$ m_ℓ, respectively (i.e., λ_j defines m_j linearly independent associated vectors for $\mathscr{A}_\infty$; see [Ga]) and

$$\delta_j := \frac{1}{2\pi i}\log\lambda_j, \quad \gamma < \Re e\ \delta_j \leq \gamma + 1\,, \quad j = 1, \ldots, \ell \tag{2.3}$$

for some $\gamma \in \mathbb{R}$.

Theorem 2.1. *Let $m = 2, \dots$, and $\mathscr{A} \in \mathscr{L}^2(\overline{\mathbb{R}})$ (or $\mathscr{A} \in \mathscr{H}_0^m(\overline{\mathbb{R}})$) an elliptic $N \times N$ matrix-valued function. Then*

$$\mathscr{A}(x) = [\mathscr{A}_-(x)]^{-1}\Xi(x)\mathscr{A}_+(x)\,, \tag{2.4}$$

$$\Xi(x) = \mathscr{A}(+\infty)\mathscr{K}\left(\frac{x-i}{x+i}\right)^{-\Delta+\varkappa} B_{\mathscr{A}_\infty}\left(\frac{1}{2\pi i}\log\frac{x-i}{x+i}\right)\mathscr{K}^{-1}\,.$$

Here:

(i) *The matrix-valued functions $\mathscr{A}_-^{\pm1}$, $\mathscr{A}_+^{\pm1}$ belong to $\mathscr{L}^1(\overline{\mathbb{R}})$ (belong to $\mathscr{H}_0^{m-1}(\overline{\mathbb{R}})$). The factors $\mathscr{A}_-^{\pm1}(x-it)$ and $\mathscr{A}_+^{\pm1}(x+it)$ have uniformly bounded analytic continuation for $t > 0$ and $\mathscr{A}_+^{\pm1}(\pm\infty) = I_N$, where I_N is the identity matrix of order N.*

(ii) *The numbers δ_j are defined in* (2.3), *the vector $\Delta := (\delta_1, \dots, \delta_\ell)$ has length N (each δ_j occurs m_j times according to its algebraic multiplicity), $\varkappa = (\varkappa_1, \dots, \varkappa_N) \in \mathbb{N}_0^N$ are integers (known as the partial indices of $\mathscr{A}$) and*

$$h^{\Delta+\varkappa} := \operatorname{diag}\{h^{\delta_1+\varkappa_1}, \dots, h^{\delta_\ell+\varkappa_N}\} \quad for \quad h \in \mathbb{C}$$

is a diagonal matrix.

(iii) *$B_{\mathscr{A}_\infty}(z)$, $z \in \mathbb{C}$, is an upper triangular polynomial matrix-valued function related to the Jordan normal form of $\mathscr{A}_\infty$, thoroughly described in Appendix,* § A.1.

Remark 2.2. The factorization (2.4) depends on a real number $\gamma \in \mathbb{R}$ (cf. (2.3)) which will be fixed uniquely later depending on the space where an operator with the symbol $\mathscr{A}$ is treated.

Proof of Theorem 2.1. Let

$$\begin{aligned} \mathscr{A}^*(x) &= (x-i)^{\Delta}B_-(x)\mathscr{A}_1(x)B_+^{-1}(x)(x+i)^{-\Delta}, \\ \mathscr{A}_1(x) &= \mathscr{K}^{-1}\mathscr{A}^{-1}(+\infty)\mathscr{A}(x)\mathscr{K}\,, \end{aligned} \tag{2.5}$$

where $B_\pm(x)$ are related to the Jordan normal form of $\mathscr{A}_\infty$ and are defined by (A.4). Due to (2.2) and (2.5), we have

$$\begin{aligned} \mathscr{A}_1(-\infty) &= \lim_{x\to-\infty}\mathscr{A}_1(x) = \mathscr{K}^{-1}[\mathscr{A}(+\infty)]^{-1}\mathscr{A}(-\infty)\mathscr{K} = \Lambda_{\mathscr{A}_\infty}B_{\mathscr{A}_\infty}(1)\,, \\ \mathscr{A}_1(+\infty) &= \lim_{x\to+\infty}\mathscr{A}_1(x) = I_N\,. \end{aligned} \tag{2.6}$$

According to their definition, the matrix-valued functions $B_\pm(x)$ and $B_{\mathscr{A}_\infty}$(x) are block-diagonal with blocks of upper triangular matrices of dimensions $m_1, \dots, m_\ell$. The matrices $\Lambda_{\mathscr{A}_\infty}$ and $(x-i)^{\pm\Delta}$ are diagonal, with blocks of equal constants (functions) of the same dimension $m_1, \dots, m_\ell$. Therefore all these matrices commute and have the following properties, cf. (A.1)–(A.5):

$$\begin{aligned} B_\pm\Lambda_{\mathscr{A}_\infty} &= \Lambda_{\mathscr{A}_\infty}B_\pm\,, \qquad (x-i)^{\pm\Delta}\Lambda_{\mathscr{A}_\infty} = \Lambda_{\mathscr{A}_\infty}(x-i)^{\pm\Delta}\,, \\ B_\pm B_{\mathscr{A}_\infty} &= B_{\mathscr{A}_\infty}B_\pm\,, \qquad B_{\mathscr{A}_\infty}(-z) = B_{\mathscr{A}_\infty}^{-1}(z)\,. \end{aligned} \tag{2.7}$$

Based on (2.6) and on (2.7) the matrix-valued function $\mathscr{A}^*$ in (2.5) can be rewritten as follows

$$\begin{aligned}
\mathscr{A}^*(x) &= \mathscr{A}_2^\pm(x) + \mathscr{A}_3^\pm(x)\,, && (2.8)\\
\mathscr{A}_2^\pm(x) &= (x-i)^\Delta B_-(x)[\mathscr{A}_1(x) - \mathscr{A}_1(\pm\infty)]B_+^{-1}(x)(x+i)^{-\Delta}\,, && (2.9)\\
\mathscr{A}_3^+(x) &= (x-i)^\Delta B_-(x)B_+^{-1}(x)(x+i)^{-\Delta}\\
&= \left(\frac{x-i}{x+i}\right)^\Delta B_{\mathscr{A}_\infty}\left(\frac{1}{2\pi i}\log\frac{x-i}{x+i}\right)\\
&= B_{\mathscr{A}_\infty}\left(\frac{1}{2\pi i}\log\frac{x-i}{x+i}\right)\left(\frac{x-i}{x+i}\right)^\Delta\,, && (2.10)\\
\mathscr{A}_3^-(x) &= (x-i)^\Delta B_-(x)\Lambda_{\mathscr{A}}B_{\mathscr{A}_\infty}(1)B_+^{-1}(x)(x+i)^{-\Delta}\\
&= \Lambda_{\mathscr{A}}B_{\mathscr{A}_\infty}(1)B_-(x)B_+^{-1}(x)(x-i)^\Delta(x+i)^{-\Delta}\\
&= \Lambda_{\mathscr{A}}B_{\mathscr{A}_\infty}(1)B_{\mathscr{A}_\infty}\left(\frac{1}{2\pi i}\log\frac{x-i}{x+i}\right)\left(\frac{x-i}{x+i}\right)^\Delta\\
&= \Lambda_{\mathscr{A}}B_{\mathscr{A}_\infty}(1)\mathscr{A}_3^+(x) = \mathscr{A}_3^+(x)B_{\mathscr{A}_\infty}(1)\Lambda_{\mathscr{A}}\,. && (2.11)
\end{aligned}$$

Further, due to the definition of the function $(x\pm i)^{\pm\Delta}$ (see (2.3), (2.5)),

$$\left(\frac{x-i}{x+i}\right)^{\pm\Delta} = \begin{cases} I_N + \mathscr{O}(\langle x\rangle^{-1}) & \text{as} \quad x\to+\infty\,,\\ \Lambda_{\mathscr{A}}^{\mp1} + \mathscr{O}(\langle x\rangle^{-1}) & \text{as} \quad x\to-\infty\,, \end{cases} \tag{2.12}$$

$$B_{\mathscr{A}_\infty}\left(\frac{1}{2\pi i}\log\frac{x-i}{x+i}\right) = \begin{cases} B_{\mathscr{A}_\infty}(1) + \mathscr{O}(\langle x\rangle^{-1}) & \text{as} \quad x\to+\infty\,,\\ B_{\mathscr{A}_\infty}(0) = I_N + \mathscr{O}(\langle x\rangle^{-1}) & \text{as} \quad x\to-\infty\,, \end{cases} \tag{2.13}$$

where $\langle x\rangle := (1+|x|^2)^{1/2}$. From (2.10)–(2.13) it is clear that $\mathscr{A}_3^\pm(-\infty) = \mathscr{A}_3^\pm(+\infty)$ and

$$\mathscr{A}_3^\pm \in \mathscr{Z}^\sigma(\mathbb{R}) \cap \mathscr{H}_0^\sigma(\mathbb{R}) \qquad \text{for all} \quad \sigma > 0\,. \tag{2.14}$$

Next we prove that

$$\mathscr{A}_2^\pm \in \mathscr{Z}^{2-\delta_0-\varepsilon}(\mathbb{R}) \quad \Big(\mathscr{A}_2^\pm \in \mathscr{H}_0^{m-\delta_0-\varepsilon}(\mathbb{R})\,,\ \text{respectively}\Big)\,, \quad 0 < \delta_0 + 2\varepsilon < 1\,, \tag{2.15}$$

where $\varepsilon > 0$ is arbitrarily small and δ_0 is defined by the relations (see (2.3))

$$0 \le \delta_0 := \max_{j,q=1,\dots,N}\{\Re e\,(\delta_j - \delta_q)\} < \delta_0 + 2\varepsilon < 1\,. \tag{2.16}$$

To this end we note that

$$\partial_x^k \mathscr{A}_1^0(x) = \mathscr{O}(\langle x\rangle^{-k-1})\,, \qquad \mathscr{A}_1^0(x) := \mathscr{A}_1(x) - \mathscr{A}_1(\pm\infty)\,, \tag{2.17}$$

where $k = 2$ in the case of $\mathscr{A} \in \mathscr{Z}^2(\overline{\mathbb{R}})$ and $k = m$ in the case of $\mathscr{A} \in \mathscr{H}_0^m(\overline{\mathbb{R}})$. A typical entry of $\mathscr{A}_2^\pm$ is

$$[\mathscr{A}_2^\pm]_{jl} = b_{jl}(x)[\mathscr{A}_1^0(x)]_{jl}\,, \quad j,l = 0,\dots,N\,,$$

with

$$b_{jl}(x) := (x-i)^{\delta_p}(x+i)^{-\delta_q}[\log(x-i)]^r[\log(x-i)]^s = \mathscr{O}\left(\langle x\rangle^{\delta_p-\delta_q+\varepsilon}\right)$$

and Propositions A.1(i), A.1(iv) with (2.12) and (2.17) yield the claimed inclusion (2.15).

From (2.8), (2.14) and (2.15) we obtain

$$\mathscr{A}^* \in \mathscr{X}^{2-\delta_0-\varepsilon}(\mathbb{R}) \qquad \left(\mathscr{A}^* \in \mathscr{H}_0^{m-\delta_0-\varepsilon}(\mathbb{R})\,,\ \text{respectively}\right)\,. \tag{2.18}$$

Then, due to Proposition A.6, the elliptic matrix-valued function $\mathscr{A}^*$ admits a classical right factorization

$$\mathscr{A}^*(x) = [\mathscr{A}_-^*(x)]^{-1}\left(\frac{x-i}{x+i}\right)^{\varkappa}\mathscr{A}_+^*(x), \tag{2.19}$$

$$\varkappa = (\varkappa_1,\ldots,\varkappa_N)\in\mathbb{Z}^N, \quad \mathbb{Z}=\{0,\pm1,\ldots\}$$

with factors $[\mathscr{A}_-^*(x)]^{\pm1}$, $[\mathscr{A}_+^*(x)]^{\pm1}$ belonging to the same algebras as $\mathscr{A}^*$ (cf. (2.18)). These factors have uniformly bounded analytic continuations into the half–planes $\Im m\ x<0$ and $\Im m\ x>0$, respectively.

Since the limits $\mathscr{A}_\pm^*(\infty)$, $\mathscr{A}^*(\infty)$ exist and $\mathscr{A}^*(\infty)=I_N$, from (2.19) there follows

$$[\mathscr{A}_-^*(\infty)]^{-1}\mathscr{A}_+^*(\infty) = \mathscr{A}^*(\infty) = I_N\,.$$

Therefore, without restricting generality we can assume that $\mathscr{A}_\pm^*(\infty)=I_N$. Then (cf. (2.19))

$$\partial_x^k\left[[\mathscr{A}_\pm^*(x)]^{\pm1}-I_N\right] = \mathscr{O}\left(\langle x\rangle^{\delta_0+\varepsilon-k-1}\right) \quad \text{as} \quad x\to\infty\,, \tag{2.20}$$

where $k=2$ in the case of $\mathscr{A}\in\mathscr{X}^2(\overline{\mathbb{R}})$ and $k=m$ in the case of $\mathscr{A}\in\mathscr{H}_0^m(\overline{\mathbb{R}})$.

From (2.5) and (2.19) we find the components of the factorization (2.4):

$$\begin{aligned}\mathscr{A}_\pm(x) :={}& \mathscr{M}_\pm(x\pm i)^{-\Delta}B_\pm^{-1}(x)\mathscr{A}_\pm^*(x)B_\pm(x)(x\pm i)^{\Delta}\mathscr{M}_\pm^{-1}\\ ={}& I_N+\mathscr{M}_\pm(x\pm i)^{-\Delta}B_\pm^{-1}(x)[\mathscr{A}_\pm^*(x)-I_N]B_\pm(x)(x\pm i)^{\Delta}\mathscr{M}_\pm^{-1}\,, \qquad (2.21)\\ \mathscr{M}_+ :={}& \mathscr{K}\,, \quad \mathscr{M}_- := \mathscr{A}(+\infty)\mathscr{K}\,.\end{aligned}$$

The theorem will be proved if we succeed in verifying the inclusions

$$\langle\cdot\rangle^k\partial^k\mathscr{A}_+^{\pm1}(\cdot),\ \ \langle\cdot\rangle^k\partial^k\mathscr{A}_-^{\pm1}(\cdot)\in\mathscr{X}^1(\mathbb{R})\,, \tag{2.22}$$

where $k=0$ in the case of $\mathscr{A}\in\mathscr{X}^2(\overline{\mathbb{R}})$ and $k=m-2$ in the case of $\mathscr{A}\in\mathscr{H}_0^m(\overline{\mathbb{R}})$.

A typical entry of the matrix $\mathscr{A}_+^{\pm1}(x)-I_N$ is

$$[\mathscr{A}_+^{\pm1}(x)-I_N]_{jq} = (x+i)^{\delta_p-\delta_r}\sum_{l\le q}c_{jql}\left[[\mathscr{A}_+^*(x)]_{jl}^{\pm1}-\delta_{jl}\right]\log^{m_{lq}}(x+i)\,,$$

$$\partial^k[\mathscr{A}_+^{\pm1}(x)-I_N]_{jq} = \begin{cases}\mathscr{O}\left(\langle x\rangle^{\Re e(\delta_p-\delta_r)+\delta_0+2\varepsilon-k-1}\right) & \text{if} \quad \Re e\ \delta_p>\Re e\ \delta_r\\ \mathscr{O}\left(\langle x\rangle^{\delta_0+2\varepsilon-k-1}\right) & \text{if} \quad \Re e\ \delta_p\le\Re e\,\delta_r\end{cases} \tag{2.23}$$

(cf. (2.20)), where $m_{qq}=0$ and δ_{jl} is the Kroneker's symbol. From (2.19) we have

$$\begin{aligned}\mathscr{A}^*_+ - \mathscr{A}^*_- \;&=\; \mathscr{A}^* - I_N + \left[I_N - \left(\frac{x-i}{x+i}\right)^{\varkappa}\right]\mathscr{A}^*_+ + [\mathscr{A}^*_- - I_N][\mathscr{A}^* - I_N]\,,\\ [\mathscr{A}^*_+]^{-1} - [\mathscr{A}^*_-]^{-1} \;&=\; I_N - [\mathscr{A}^*]^{-1} + \left[\left(\frac{x-i}{x+i}\right)^{-\varkappa} - I_N\right][\mathscr{A}^*_-]^{-1}\\ &\qquad + \left[[\mathscr{A}^*_-]^{-1} - I_N\right]\left[I_N - [\mathscr{A}^*]^{-1}\right]\,,\end{aligned} \tag{2.24}$$

and applying (2.18) and (2.20) we obtain

$$\begin{aligned}\partial^k_x\left[[\mathscr{A}^*_+]^{\pm1}(x) - [\mathscr{A}^*_-]^{\pm1}(x)\right]_{jl} \;&=\; \mathscr{O}(\langle x\rangle^{-k-1}) + \partial^k_x\sum_{r=1}^{N}\{[\mathscr{A}^*_-]^{\pm1}_{jr} - \delta_{jr}\}[\mathscr{A}^* - I_N]_{rl}\\ &\qquad +\mathscr{O}\left(\langle x\rangle^{\Re e\,(\delta_j-\delta_l)+\varepsilon-k-1}\right)\\ &=\; \sum_{r=1}^{N}\mathscr{O}\left(\langle x\rangle^{\Re e\,(\delta_r-\delta_l)+\varepsilon-1-k}\right) = \mathscr{O}\left(\langle x\rangle^{\delta^+_l+\varepsilon-1-k}\right)\end{aligned}$$

where $\delta^+_j \;:=\; \max\limits_q\{\Re e\,[\delta_q - \delta_j]\} \;=\; \Re e\,[\delta_{j_*} - \delta_j]$ for a certain $1 \le j_* \le n$ (note that we have inserted $\partial^l_x\{[\mathscr{A}^*_-]^{\pm1}_{jr}(x) - \delta_{jl}\} = \mathscr{O}(\langle x\rangle^{\delta_0+\varepsilon-l-1}) = \mathscr{O}(\langle x\rangle^{-l})$; cf. (2.20)). According to Proposition A.1(i), $\left[[\mathscr{A}^*_+]^{\pm1} - [\mathscr{A}^*_-]^{\pm1}\right]_{jl} \in \widetilde{\mathscr{H}}^{k-\delta^+_l-\varepsilon}(\mathbb{R})$ (we remind that $k=0$ in the case of $\mathscr{A} \in \mathscr{Z}^2(\overline{\mathbb{R}})$ and $k=m-2$ in the case of $\mathscr{A} \in \mathscr{H}^m_0(\overline{\mathbb{R}})$).

We will use the Hilbert transformation $H_{\mathbb{R}}$ (cf. (A.16)) to define the projections $P^{\pm}_{\mathbb{R}} = \dfrac{1}{2}(I \pm H_{\mathbb{R}})$ that eliminate functions, analytic in the half–planes $\mp\Im m\ x < 0$ (see [ClGo, GoKr]), and are bounded in $\widetilde{\mathscr{H}}^{\mu}(\mathbb{R})$ (see Theorem A.5); hence

$$[\mathscr{A}^*_\pm]^{\pm1}_{jl} = \pm P^{\pm}_{\mathbb{R}}\left[[\mathscr{A}^*_+]^{\pm1} - [\mathscr{A}^*_-]^{\pm1}\right]_{jl} \in \widetilde{\mathscr{H}}^{k-\delta^+_l-\varepsilon}(\mathbb{R})$$

and, therefore (see (A.9) and cf. (2.20)),

$$\partial^k_x\left[[\mathscr{A}^*_\pm]^{\pm1}_{jl}(x) - I_N\right]_{jl} = \mathscr{O}\left(\langle x\rangle^{\delta^+_l+\varepsilon-k-1}\right)\,. \tag{2.25}$$

Inserting the obtained asymptotic for $\left[[\mathscr{A}^*_-]^{\pm1}_{jl}(x) - I_N\right]_{jl}$ into (2.24) and invoking (2.18) once again we get a more precise asymptotical behavior

$$\begin{aligned}\partial^k_x&\left[[\mathscr{A}^*_+]^{\pm1}(x) - [\mathscr{A}^*_-]^{\pm1}(x)\right]_{jl} = \mathscr{O}\left(\langle x\rangle^{-k-1}\right)\\ &+\sum_{r=1}^{N}\mathscr{O}\left(\langle x\rangle^{\delta^+_r+2\varepsilon-1+\Re e\,(\delta_r-\delta_l)+\varepsilon-k-1}\right) + \mathscr{O}\left(\langle x\rangle^{\Re e\,(\delta_j-\delta_l)+\varepsilon-k-1}\right)\\ &=\sum_{r=1}^{N}\mathscr{O}\left(\langle x\rangle^{\Re e\,(\delta_{r_*}-\delta_l)+3\varepsilon-k-2}\right) + \mathscr{O}\left(\langle x\rangle^{\Re e\,(\delta_j-\delta_l)+\varepsilon-k-1}\right)\\ &=\mathscr{O}\left(\langle x\rangle^{\Re e\,(\delta_j-\delta_l)+\varepsilon-k-1}\right)\,,\end{aligned}$$

where $\delta_{r_*} := \delta_r + \delta_r^+$. Thus, $\left[[\mathscr{A}_+^*]^{\pm 1} - [\mathscr{A}_-^*]^{\pm 1}\right]_{jl} \in \widetilde{\mathscr{H}}^{k-\Re e\,(\delta_j-\delta_l)-\varepsilon}(\mathbb{R})$ and we conclude, as above, $[\mathscr{A}_\pm^*]_{jl}^{\pm 1} = \pm P_{\mathbb{R}}^{\pm}\left[[\mathscr{A}_+^*]^{\pm 1} - [\mathscr{A}_-^*]^{\pm 1}\right]_{jl} \in \widetilde{\mathscr{H}}^{k-\Re e\,(\delta_j-\delta_l)-\varepsilon}(\mathbb{R})$. The latter yields (cf. (2.20))

$$\partial_x^k\left[[\mathscr{A}_\pm^*(x)]_{jl}^{\pm 1} - I_N\right]_{jl} = \mathscr{O}\left(\langle x\rangle^{\Re e\,(\delta_j-\delta_l)+\varepsilon-k-1}\right).$$

By virtue of (2.23)

$$\partial_x^k[\mathscr{A}_+]_{jq}^{\pm 1}(x) = \mathscr{O}\left(\langle x\rangle^{\Re e\,(\delta_q-\delta_j)+\Re e\,(\delta_j-\delta_l)+2\varepsilon-k-1}\right) = \mathscr{O}(\langle x\rangle^{\theta-k-1})$$

since $\delta_l = \delta_j$ and since $\Re e\,(\delta_q-\delta_j)+2\varepsilon = \theta < 1$. □

For further purposes, we recall that a matrix $\mathscr{B}$ is called *normal* if it commutes with its own transposed matrix $\mathscr{B}^\top\mathscr{B} = \mathscr{B}\mathscr{B}^\top$. Moreover, a matrix $\mathscr{B}$ is called *positive definite* if

$$(\mathscr{B}\eta,\eta) \geq M|\eta|^2 \qquad \forall\,\eta\in\mathbb{C}^n$$

with some constant $M > 0$.

Lemma 2.3. *If the matrix $\mathscr{A}_\infty$ in* (2.2) *is normal, then it is simple $\ell = N$ (i.e., each eigenvalue λ_j has algebraic multiplicity 1) and, therefore, $\mathscr{A}_\infty$ is diagonalizable:*

$$B_{\mathscr{A}_\infty}(x) \equiv I\,, \quad \mathscr{A}_\infty = \mathscr{K}\,\mathrm{diag}\,\{\lambda_1,\dots,\lambda_N\}\,\mathscr{K}^*, \tag{2.26}$$

$$\det\mathscr{K} \neq 0, \qquad \mathscr{K}^{-1} = \mathscr{K}^*.$$

If the matrices $\mathscr{A}(\pm\infty)$ are positive definite, then $\mathscr{A}_\infty$ in (2.2) *is simple, the eigenvalues $\lambda_1,\dots,\lambda_\ell$ are all real positive numbers and, therefore,*

$$\Re e\,\delta_1 = \cdots = \Re e\,\delta_\ell \equiv 0^1\,. \tag{2.27}$$

Proof. For the first claim of the lemma we quote [La, Theorem 2.10.2].

The second assertion is proved in [DuSäWe, Lemma A.6] as follows. Since the matrices $\mathscr{A}(\pm\infty)$ are positive definite, the square roots $[\mathscr{A}(+\infty)]^{\pm 1/2}$ are well defined and the matrix

$$\begin{aligned}\mathscr{A}_1(\omega) &:= [\mathscr{A}(+\infty)]^{1/2}\,\mathscr{A}_\infty(\omega)\,[\mathscr{A}(+\infty)]^{-1/2}\\ &= [\mathscr{A}(+\infty)]^{-1/2}\,\mathscr{A}(-\infty)\,[\mathscr{A}(+\infty)]^{-1/2},\end{aligned}$$

due to similarity, has the common eigenvalues, the common eigenvectors and the common Jordan chains of associated vectors with $\mathscr{A}_\infty$. On the other hand $\mathscr{A}_1$ is self-adjoint, i.e., is normal and has no associated vectors as noted above. Let $\eta,\dots,\eta_N\in\mathbb{C}^N$ be eigenvectors corresponding to the eigenvalues $\lambda_1,\dots,\lambda_N$; then

$$\mathscr{A}_\infty\eta_j = \lambda_j\eta_j, \quad j = 1,\dots,N$$

and we get

$$\lambda_j = \frac{(\mathscr{A}_\infty(+\infty)\eta_j,\eta_j)}{(\mathscr{A}_\infty(-\infty)\eta_j,\eta_j)} > 0$$

because of the positive definiteness of $\mathscr{A}(\pm\infty)$. This implies (2.27). □

---

[1] The numbers δ_j in (2.3) and ν_j in [DuSäWe, (A.32)] are related as follows: $\delta_j = -i\nu_j$.

3. Asymmetric and Anti-Symmetric Factorizations

In this section we present two different kinds of factorizations of matrix-valued functions which display some symmetries in their structure. These factorizations are tightly connected with the theory of *convolution type operators with symmetry* [CaSpTe2]

$$T = r_+\mathscr{F}^{-1}\mathscr{B}\cdot\mathscr{F}\ell^c : [L^2(\mathbb{R}_+)]^N \to [L^2(\mathbb{R}_+)]^N\,, \tag{3.1}$$

and play a central role in the description of (generalized) invertibility properties of such operators (cf. [CaSp, CaSpTe2]). Here, the operator r_+ stands for the restriction to the positive half-line, $\mathscr{F}^{-1}$ and $\mathscr{F}$ are the inverse and direct Fourier transformations, $\mathscr{B}$ is a measurable $N \times N$ matrix-valued function, and ℓ^c denotes the even (ℓ^e) or odd (ℓ^o) extension as a continuous operator from $[L^2(\mathbb{R}_+)]^N$ into $[L^2(\mathbb{R})]^N$.

We shall also make use of $[L^2_\pm(\mathbb{R})]^{N\times N}$ to be the images of the space $[L^2(\mathbb{R})]^{N\times N}$ under the the projections

$$P^\pm_{\mathbb{R}} = \frac{1}{2}(I \pm H_{\mathbb{R}})\,. \tag{3.2}$$

For a subspace $[X(\mathbb{R})]^{N\times N}$ of $[L^2(\mathbb{R})]^{N\times N}$, and a weight function ρ, the notation $[X(\mathbb{R},\rho)]^{N\times N}$ is used for the subspace of those elements $\mathscr{B}$ for which $\rho\mathscr{B} \in [X(\mathbb{R})]^{N\times N}$. Additionally, we will make use of the subspaces

$$\begin{aligned} [L^{2,e}(\mathbb{R},\rho)]^{N\times N} &= \left\{\mathscr{B} \in [L^2(\mathbb{R},\rho)]^{N\times N} \;:\; \mathscr{B}(x) = \mathscr{B}(-x)\right\} \\ [L^{2,o}(\mathbb{R},\rho)]^{N\times N} &= \left\{\mathscr{B} \in [L^2(\mathbb{R},\rho)]^{N\times N} \;:\; \mathscr{B}(x) = -\mathscr{B}(-x)\right\}\,. \end{aligned}$$

Definition 3.1. A matrix-valued function $\mathscr{B} \in \mathscr{G}[L^\infty(\mathbb{R})]^{N\times N}$ (i.e., $\mathscr{B}$ is invertible in $[L^\infty(\mathbb{R})]^{N\times N}$) admits an *asymmetric generalized factorization with respect to L^2 and ℓ^e*, written as

$$\mathscr{B}(x) \;=\; \mathscr{B}_-(x)\left(\frac{x-i}{x+i}\right)^\kappa \mathscr{B}_e(x)\,, \qquad x \in \mathbb{R}\,, \tag{3.3}$$

if $\kappa = (\kappa_1,\dots,\kappa_N)$, with $\kappa_1,\dots,\kappa_N \in \mathbb{Z}$,

$$\mathscr{B}_- \in [L^2_-(\mathbb{R},\lambda_-^{-2})]^{N\times N}\,, \qquad \mathscr{B}_-^{-1} \in [L^2_-(\mathbb{R},\lambda_-^{-1})]^{N\times N}\,, \tag{3.4}$$

$$\mathscr{B}_e \in [L^{2,e}(\mathbb{R},\lambda^{-1})]^{N\times N}\,, \qquad \mathscr{B}_e^{-1} \in [L^{2,e}(\mathbb{R},\lambda^{-2})]^{N\times N} \tag{3.5}$$

and if

$$V_e = A_e^{-1}\ell^e r_+ A_-^{-1} \tag{3.6}$$

is an operator defined on a dense subspace of $[L^2(\mathbb{R})]^m$ possessing a bounded extension to $[L^2(\mathbb{R})]^m$, with

$$A_e = \mathscr{F}^{-1}\mathscr{B}_e\cdot\mathscr{F}\,, \tag{3.7}$$

$$A_- = \mathscr{F}^{-1}\mathscr{B}_-\cdot\mathscr{F}\,. \tag{3.8}$$

As usual, the factor spaces (where the factors of $\mathscr{B}$ can be found) are the closures of the spaces of bounded rational functions without poles in the closed lower half–plane $\overline{\mathbb{C}_-} = \{\xi \in \mathbb{C} : \Im m\ \xi \leq 0\}$ or of those which are even, respectively, due to the weighted L^2 norm.

When all κ_j components of κ in (3.3) are zero, we will denote the factorization as a *canonical asymmetric generalized factorization with respect to* L^2 *and* ℓ^e and so we shall use the word *canonical* as in other kinds of factorizations.

Definition 3.2. We will say that a matrix-valued function $\mathscr{B} \in \mathscr{G}[L^\infty(\mathbb{R})]^{N\times N}$ admits an *asymmetric generalized factorization with respect to* L^2 *and* ℓ^o, if it is factorable in the form of (3.3), with $\kappa = (\kappa_1, \ldots, \kappa_N)$, $\kappa_1, \ldots, \kappa_N \in \mathbb{Z}$,

$$\mathscr{B}_- \in [L^2(\mathbb{R}, \lambda_-^{-1})]^{N\times N}, \qquad \mathscr{B}_-^{-1} \in [L^2_-(\mathbb{R}, \lambda_-^{-2})]^{N\times N}, \tag{3.9}$$

$$\mathscr{B}_e \in [L^{2,e}(\mathbb{R}, \lambda^{-2})]^{N\times N}, \qquad \mathscr{B}_e^{-1} \in [L^{2,e}(\mathbb{R}, \lambda^{-1})]^{N\times N} \tag{3.10}$$

and if

$$V_o = A_e^{-1} \ell^o r_+ A_-^{-1} \tag{3.11}$$

is an operator defined on a dense subspace of $[L^2(\mathbb{R})]^N$ possessing a bounded extension to $[L^2(\mathbb{R})]^N$, and with A_e and A_- as in (3.7) and (3.8), respectively.

Given a matrix-valued function $\mathscr{A}$, on the real line, we will abbreviate by $\widetilde{\mathscr{A}}$ that one defined by

$$\widetilde{\mathscr{A}}(x) = \mathscr{A}(-x), \qquad x \in \mathbb{R}. \tag{3.12}$$

Definition 3.3. A matrix-valued function $\mathscr{C} \in \mathscr{G}[L^\infty(\mathbb{R})]^{N\times N}$ admits an *anti-symmetric generalized factorization with respect to* L^2 *and* ℓ^e

$$\mathscr{C}(x) = \mathscr{C}_-(x) \left(\frac{x-i}{x+i}\right)^{2\kappa} \widetilde{\mathscr{C}}_-^{-1}(x), \qquad x \in \mathbb{R}, \tag{3.13}$$

with integer valued partial indices $\kappa = (\kappa_1, \ldots, \kappa_N)$, $\kappa_1, \ldots, \kappa_N \in \mathbb{Z}$, if:

(i) the factors belong to the following spaces

$$\mathscr{C}_- \in [L^2_-(\mathbb{R}, \lambda_-^{-2})]^{N\times N}, \qquad \mathscr{C}_-^{-1} \in [L^2_-(\mathbb{R}, \lambda_-^{-1})]^{N\times N}; \tag{3.14}$$

(ii) the operator

$$U_e = \widetilde{A}_- \ell^e r_+ A_-^{-1} \tag{3.15}$$

defined on a dense subset of $[L^2(\mathbb{R})]^N$ has a bounded extension to $[L^2(\mathbb{R})]^N$ (where $\widetilde{A}_- = \mathscr{F}^{-1}\widetilde{\mathscr{C}}_- \cdot \mathscr{F}$ and $A_- = \mathscr{F}^{-1}\mathscr{C}_- \cdot \mathscr{F}$).

Definition 3.4. We will say that a matrix-valued function $\mathscr{C} \in \mathscr{G}[L^\infty(\mathbb{R})]^{N\times N}$ admits an *anti-symmetric generalized factorization with respect to* L^2 *and* ℓ^o, if:

(i) $\mathscr{C}$ is decomposed as in (3.13) with integer valued partial indices $\kappa = (\kappa_1, \ldots, \kappa_N)$, $\kappa_1, \ldots, \kappa_N \in \mathbb{Z}$;

(ii) the factors belong to the following spaces

$$\mathscr{C}_- \in [L^2_-(\mathbb{R}, \lambda_-^{-1})]^{N\times N}, \qquad \mathscr{C}_-^{-1} \in [L^2_-(\mathbb{R}, \lambda_-^{-2})]^{N\times N};$$

(iii) the operator

$$U_o = \widetilde{A}_- \ell^o r_+ A_-^{-1} \tag{3.16}$$

defined on a dense subset of $[L^2(\mathbb{R})]^N$ has a bounded extension to $[L^2(\mathbb{R})]^N$ (where $\widetilde{A}_- = \mathscr{F}^{-1}\widetilde{\mathscr{C}}_- \cdot \mathscr{F}$ and $A_- = \mathscr{F}^{-1}\mathscr{C}_- \cdot \mathscr{F}$).

In the next result we will explore a link between asymmetric and anti-symmetric generalized factorizations, which is useful for transferring results between the two types of factorizations.

Lemma 3.5. *Let* $\mathscr{B} \in \mathscr{G}[L^\infty(\mathbb{R})]^{N\times N}$ *and consider* $\mathscr{C} = \mathscr{B}\widetilde{\mathscr{B}}^{-1}$.

(i) *If* $\mathscr{B}$ *admits an asymmetric generalized factorization with respect to* L^2 *and* ℓ^c,

$$\mathscr{B}(x) \;=\; \mathscr{B}_-(x)\left(\frac{x-i}{x+i}\right)^{\kappa}\mathscr{B}_e(x)\,, \qquad x \in \mathbb{R}\,, \tag{3.17}$$

then $\mathscr{C}$ *admits an anti-symmetric generalized factorization with respect to* L^2 *and* ℓ^c *in the form*

$$\mathscr{C}(x) = \mathscr{B}_-(x)\left(\frac{x-i}{x+i}\right)^{2\kappa}\widetilde{\mathscr{B}}_-^{-1}(x)\,, \qquad x \in \mathbb{R}\,. \tag{3.18}$$

(ii) *If* $\mathscr{C}$ *admits an anti-symmetric generalized factorization with respect to* L^2 *and* ℓ^c,

$$\mathscr{C}(x) = \mathscr{C}_-(x)\left(\frac{x-i}{x+i}\right)^{2\kappa}\widetilde{\mathscr{C}}_-^{-1}(x)\,, \qquad x \in \mathbb{R}\,, \tag{3.19}$$

then $\mathscr{B}$ *admits an asymmetric generalized factorization with respect to* L^2 *and* ℓ^c *in the form*

$$\mathscr{B}(x) = \mathscr{C}_-(x)\left(\frac{x-i}{x+i}\right)^{\kappa}\left(\left(\frac{x-i}{x+i}\right)^{-\kappa}\mathscr{C}_-^{-1}(x)\mathscr{B}(x)\right)\,, \qquad x \in \mathbb{R}\,, \tag{3.20}$$

where $\left(\frac{x-i}{x+i}\right)^{-\kappa}\mathscr{C}_-^{-1}(x)\mathscr{B}(x)$ *is the even factor (cf.* (3.3)*).*

Proof. We will present the proof for $\ell^c = \ell^e$. The case $\ell^c = \ell^o$ runs analogously, with obvious changes.

(i) Assuming an asymmetric generalized factorization with respect to L^2 and ℓ^e for $\mathscr{B}$,

$$\mathscr{B}(x) = \mathscr{B}_-(x)\left(\frac{x-i}{x+i}\right)^{\kappa}\mathscr{B}_e(x)\,, \qquad x \in \mathbb{R}\,, \tag{3.21}$$

with $\kappa_j \in \mathbb{Z}$, $j = 1,\dots,N$, $\mathscr{B}_- \in [L^2_-(\mathbb{R},\lambda_-^{-2})]^{N\times N}$, $\mathscr{B}_-^{-1} \in [L^2_-(\mathbb{R},\lambda_-^{-1})]^{N\times N}$, $\mathscr{B}_e \in [L^{2,e}(\mathbb{R},\lambda^{-1})]^{N\times N}$, $\mathscr{B}_e^{-1} \in [L^{2,e}(\mathbb{R},\lambda^{-2})]^{N\times N}$ and where

$$V_e = \mathscr{F}^{-1}\mathscr{B}_e^{-1} \cdot \mathscr{F}\ell^e r_+ \mathscr{F}^{-1}\mathscr{B}_-^{-1} \cdot \mathscr{F} \tag{3.22}$$

is an operator defined on a dense subspace of $[L^2(\mathbb{R})]^N$ possessing a bounded extension to $[L^2(\mathbb{R})]^N$, we start by choosing the same "minus" factor $\mathscr{B}_-$ for the factorization of $\mathscr{C}$ and observe in addition that

$$\widetilde{\mathscr{B}}^{-1}(x) = \mathscr{B}_e^{-1}(x)\left(\frac{x-i}{x+i}\right)^{\kappa} \widetilde{\mathscr{B}}_-^{-1}(x) \tag{3.23}$$

holds due to the even property of $\mathscr{B}_e$. Therefore,

$$\begin{aligned}\mathscr{C}(x) = \mathscr{B}(x)\,\widetilde{\mathscr{B}}^{-1}(x) \;&=\; \left(\mathscr{B}_-(x)\left(\frac{x-i}{x+i}\right)^{\kappa}\mathscr{B}_e(x)\right)\left(\mathscr{B}_e^{-1}(x)\left(\frac{x-i}{x+i}\right)^{\kappa}\widetilde{\mathscr{B}}_-^{-1}(x)\right)\\ &=\; \mathscr{B}_-(x)\left(\frac{x-i}{x+i}\right)^{2\kappa}\widetilde{\mathscr{B}}_-^{-1}(x)\,,\end{aligned} \tag{3.24}$$

with

$$\mathscr{B}_- \in [L^2_-(\mathbb{R},\lambda_-^{-2})]^{N\times N}\,, \qquad \mathscr{B}_-^{-1} \in [L^2_-(\mathbb{R},\lambda_-^{-1})]^{N\times N}\,, \tag{3.25}$$

or equivalently

$$\widetilde{\mathscr{B}}_- \in [L^2_+(\mathbb{R},\lambda_+^{-2})]^{N\times N}\,, \qquad \widetilde{\mathscr{B}}_-^{-1} \;\in\; [L^2_+(\mathbb{R},\lambda_+^{-1})]^{N\times N}\,. \tag{3.26}$$

The supposition of having an asymmetric generalized factorization includes that

$$V = \mathscr{F}^{-1}\mathscr{B}_e^{-1}\cdot\mathscr{F}\ell^e r_+\mathscr{F}^{-1}\mathscr{B}_-^{-1}\cdot\mathscr{F} \tag{3.27}$$

is a bounded operator (densely defined) in $[L^2(\mathbb{R})]^N$. As in the theory of generalized factorizations [Kr, § 9], this last condition (3.27) can be equivalently replaced by others. In particular, together with (3.23) we obtain that

$$U_e = \mathscr{F}^{-1}\widetilde{\mathscr{B}}_-\cdot\mathscr{F}\ell^e r_+\mathscr{F}^{-1}\mathscr{B}_-^{-1}\cdot\mathscr{F} \tag{3.28}$$

is a bounded operator also (densely defined) in $[L^2(\mathbb{R})]^N$.

(ii) If $\mathscr{C}$ admits an anti-symmetric generalized factorization with respect to L^2 and ℓ^e,

$$\mathscr{C}(x) = \mathscr{B}(x)\,\widetilde{\mathscr{B}}^{-1}(x) = \mathscr{C}_-(x)\left(\frac{x-i}{x+i}\right)^{2\kappa}\widetilde{\mathscr{C}}_-^{-1}(x)\,, \qquad x\in\mathbb{R}\,, \tag{3.29}$$

then choosing

$$\mathscr{B}_e(x) = \left(\frac{x-i}{x+i}\right)^{-\kappa}\mathscr{C}_-^{-1}(x)\,\mathscr{B}(x) \tag{3.30}$$

$$\mathscr{B}_-(x) = \mathscr{C}_-(x) \tag{3.31}$$

it directly follows that

$$\mathscr{B}(x) = \mathscr{B}_-(x)\left(\frac{x-i}{x+i}\right)^{\kappa}\mathscr{B}_e(x)\,, \qquad x\in\mathbb{R}\,. \tag{3.32}$$

In addition, due to (3.29), we have

$$\mathscr{C}_-^{-1}(x)\,\mathscr{B}(x)\,\widetilde{\mathscr{B}}^{-1}(x) = \left(\frac{x-i}{x+i}\right)^{2\kappa} \widetilde{\mathscr{C}}_-^{-1}(x) \tag{3.33}$$

$$\widetilde{\mathscr{C}}_-^{-1}(x)\,\widetilde{\mathscr{B}}(x) = \left(\frac{x-i}{x+i}\right)^{-2\kappa} \mathscr{C}_-^{-1}(x)\,\mathscr{B}(x)\,, \tag{3.34}$$

and therefore (cf. (3.30) and the first identity in (3.29))

$$\widetilde{\mathscr{B}_e}(x) = \left(\frac{x-i}{x+i}\right)^{\kappa} \widetilde{\mathscr{C}}_-^{-1}(x)\,\widetilde{\mathscr{B}}(x) = \left(\frac{x-i}{x+i}\right)^{-\kappa} \mathscr{C}_-^{-1}(x)\mathscr{B}(x) = \mathscr{B}_e(x)\,, \tag{3.35}$$

which in particular shows that $\mathscr{B}_e$ is an even function.

Now, due to the anti-symmetric generalized factorization of $\mathscr{C}$, we already know that

$$\mathscr{B}_- = \mathscr{C}_- \in [L_-^2(\mathbb{R}, \lambda_-^{-2})]^{N\times N}\,, \qquad \mathscr{B}_-^{-1} = \mathscr{C}_-^{-1} \in [L_-^2(\mathbb{R}, \lambda_-^{-1})]^{N\times N} \tag{3.36}$$

which together with the fact that $\mathscr{B} \in \mathscr{G}[L^\infty(\mathbb{R})]^{N\times N}$, and the form of the even function $\mathscr{B}_e$ in (3.30) leads to

$$\mathscr{B}_e \in [L^{2,e}(\mathbb{R}, \lambda^{-1})]^{N\times N}\,, \qquad \mathscr{B}_e^{-1} \in [L^2(\mathbb{R}, \lambda^{-2})]^{N\times N}\,. \tag{3.37}$$

Finally, similarly as in part (i), we obtain that

$$V_e = \mathscr{F}^{-1}\mathscr{B}_e^{-1} \cdot \mathscr{F}\ell^e r_+ \mathscr{F}^{-1}\mathscr{B}_-^{-1} \cdot \mathscr{F} \tag{3.38}$$

is bounded in $[L^2(\mathbb{R})]^N$ (as an extended operator from a dense subspace). ☐

Theorem 3.6. *Let $m = 2, \ldots$, $\mathscr{C} \in \mathscr{Z}^2(\overline{\mathbb{R}})$ (or $\mathscr{C} \in \mathscr{H}_0^m(\overline{\mathbb{R}})$) be a $N \times N$ elliptic matrix-valued function and*

$$\mathscr{C}_\infty := [\mathscr{C}(+\infty)]^{-1}\mathscr{C}(-\infty)\,. \tag{3.39}$$

Let $\lambda_1, \ldots, \lambda_\ell$ be all eigenvalues with Riesz indices $m_1, \cdots, m_\ell$ of the matrix $\mathscr{C}_\infty$, and consider the Jordan normal decomposition of $\mathscr{C}_\infty$,

$$\mathscr{C}_\infty = \mathscr{K}\Lambda_{\mathscr{C}_\infty} B_{\mathscr{C}_\infty}(1)\mathscr{K}^{-1} \tag{3.40}$$

(cf. the Appendix A.1 *for details). Further, let*

$$\delta_j := \frac{1}{2\pi i}\log\lambda_j, \quad \gamma < \Re e\ \delta_j \le \gamma + 1\,, \quad j = 1, \ldots, \ell \tag{3.41}$$

for some $\gamma \in \mathbb{R}$, and consider

$$\begin{aligned} \mathscr{C}^*(x) &:= (x-i)^{\Delta} B_-(x)\mathscr{C}_1(x)B_+^{-1}(x)(x+i)^{-\Delta}, \\ \mathscr{C}_1(x) &:= \mathscr{K}^{-1}\mathscr{C}^{-1}(+\infty)\mathscr{C}(x)\mathscr{K}\,, \end{aligned} \tag{3.42}$$

with $\Delta = (\delta_1, \ldots, \delta_\ell)$ having length N (where each δ_j occurs m_j times according to its algebraic multiplicity) and $B_\pm(x)$ are related to the Jordan normal form of $\mathscr{C}_\infty$ (cf. (A.4)*).*

If $\mathscr{C}^*$ admits an anti-symmetric factorization (within the later mentioned classes),

$$\mathscr{C}^*(x) = \mathscr{C}^*_-(x)\left(\frac{x-i}{x+i}\right)^{2\varkappa}\left[\widetilde{\mathscr{C}^*_-}(x)\right]^{-1}, \qquad (3.43)$$
$$\varkappa = (\varkappa_1, \dots, \varkappa_N) \in \mathbb{Z}^N,$$

then the initial matrix $\mathscr{C}$ admits the factorization

$$\mathscr{C}(x) = \mathscr{C}_-(x)\,\Xi(x)\,\left[\widetilde{\mathscr{C}_-}(x)\right]^{-1}, \qquad (3.44)$$

$$\Xi(x) = \mathscr{C}(+\infty)\mathscr{K}\left(\frac{x-i}{x+i}\right)^{-\Delta+2\varkappa} B^{-1}_{\mathscr{C}_\infty}\left(\frac{1}{2\pi i}\log\frac{x-i}{x+i}\right)\mathscr{K}^{-1},$$

where the matrix-valued functions $\mathscr{C}^{\pm 1}_-$ belong to $\mathscr{L}^1(\mathbb{R})$ (or belong to $\mathscr{H}^{m-1}_0(\mathbb{R})$, respectively), and $\mathscr{C}^{\pm 1}_-(x - it)$ have uniformly bounded analytic continuation for $t > 0$.

The proof of Theorem 3.6 runs analogously as the proof of Theorem 2.1, considering only obvious differences due to the present symmetries of the matrices. The proof is therefore omitted in here.

This last result (together with Lemma 3.5) can be used in the description of the (generalized) inverses of convolution type operator with symmetry T (introduced in (3.1)), as described in [CaSpTe2, Theorem 3.2].

A. Appendix

In the Appendix we have collected related results which either are known and are applied in the foregoing sections, or might be useful for further considerations. In our exposition we follow mostly [ChDu, §§ 1.6–1.7].

A.1. Jordan Decomposition

Let $\mathscr{B}$ be an elliptic $N \times N$ matrix ($\det \mathscr{B} \neq 0$) and $\lambda_1, \dots, \lambda_\ell$ be the eigenvalues of $\mathscr{B}$ with algebraic multiplicities $m_1, \dots, m_\ell$, respectively. Hence the length of the chain of associated vectors with the eigenvalue λ_j is $\sum\limits_{j=1}^{\ell} m_j = N$. Then $\mathscr{B}$ has the following decompositions

$$\begin{aligned} &\mathscr{B} = \mathscr{K}_0 \mathscr{J}_{\mathscr{B}} \mathscr{K}_0^{-1} = \mathscr{K}\Lambda_{\mathscr{B}} B_{\mathscr{B}}(1)\mathscr{K}^{-1}, \\ &\det\mathscr{K}_0 \neq 0, \quad \det\mathscr{K} \neq 0, \end{aligned} \qquad (A.1)$$

where the matrices $B_{\mathscr{B}}$ and $\mathscr{J}_{\mathscr{B}}$ are quasi–diagonal

$$\begin{aligned} \mathscr{J}_{\mathscr{B}} &:= \Lambda_{\mathscr{B}} + H_{\mathscr{B}} = \operatorname{diag}\{\lambda_1 I_{m_1} + H_{m_1}, \dots, \lambda_\ell I_{m_\ell} + H_{m_\ell}\} \\ B_{\mathscr{B}}(x) &:= \operatorname{diag}\{B_{m_1}(x), \dots, B_{m_\ell}(x)\}, \quad x \in \mathbb{C}, \\ B_m(z) &:= \exp(zH_m), \qquad z \in \mathbb{C}, \\ \Lambda_{\mathscr{B}} &:= \operatorname{diag}\{\lambda_1 I_{m_1}, \dots, \lambda_\ell I_{m_\ell}\}, \\ H_{\mathscr{B}} &:= \operatorname{diag}\{H_{m_1}, \dots, H_{m_\ell}\}; \end{aligned}$$

I_m is the identity and H_m is a nilpotent matrix that satisfies $H_m^m = 0$:

$$I_m := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}_{m\times m}, \quad H_m := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{m\times m}.$$

The first representation in (A.1) is known as the Jordan normal form and $\lambda I_m + H_m$ is the Jordan cell of dimension m

$$\lambda I_m + H_m = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}_{m\times m}.$$

Since $B_m(z) = \exp(zH_m),\ z \in \mathbb{C}$, and H_m is nilpotent, the exponent has a finite expansion

$$\begin{aligned} B_m(z) \quad &:= \quad \exp(zH_m) := I_n + \sum_{k=1}^{m-1} \frac{z^k}{k!} H_m^k \\ &= \begin{pmatrix} 1 & \frac{z}{1!} & \frac{z^2}{2!} & \cdots & \frac{z^{m-2}}{(m-2)!} & \frac{z^{m-1}}{(m-1)!} \\ 0 & 1 & \frac{z}{1!} & \cdots & \frac{z^{m-3}}{(m-3)!} & \frac{z^{m-2}}{(m-2)!} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 & \frac{z}{1!} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}_{m\times m}, \qquad z \in \mathbb{C}. \end{aligned}$$

The sets $\{B_{\mathscr{B}}(z)\}_{z\in\mathbb{C}}$ and $\{B_m(z)\}_{z\in\mathbb{C}}$ are one parameter groups (see [Ar, §§ 14–23]) of matrix operators, and have the following properties:

$$\begin{aligned} &B_{\mathscr{B}}(z_1 + z_2) = B_{\mathscr{B}}(z_1)B_{\mathscr{B}}(z_2), \\ &B_{\mathscr{B}}(0) = I_N, \qquad B_{\mathscr{B}}(-z) = [B_{\mathscr{B}}(z)]^{-1}, \\ &[B_{\mathscr{B}}(z)]^\gamma := \exp(z\gamma H_{\mathscr{B}}) = B_{\mathscr{B}}(\gamma z), \qquad z, \gamma \in \mathbb{C}. \end{aligned} \tag{A.2}$$

According to the definition, e.g., in [Ga, § V.1]

$$b = \frac{1}{2\pi i} \log \mathscr{B} := \frac{1}{(2\pi)^2} \int_\Gamma [\mathscr{B} - zI]^{-1} \log z\, dz,$$

where I is the identity matrix, Γ is a closed contour, surrounding all eigenvalues $\lambda_1, \ldots, \lambda_\ell$ of $\mathscr{B}$ and leaving outside the negative real half–axes $\Re e\ z \le 0$. We assume $\log z := \log|z| + i\mathrm{Arg}z, \quad -\pi < \mathrm{Arg}z < \pi$.

Here is the "purely algebraic" definition of the above presented logarithm:

$$\begin{aligned} b = \frac{1}{2\pi i}\log \mathscr{B} \ &:= \ \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\mathscr{B}^k = \frac{1}{2\pi i}\mathscr{K}\log\left[\Lambda_{\mathscr{B}}B_{\mathscr{B}}(1)\right]\mathscr{K}^{-1} \\ &= \ \mathscr{K}\left\{\Delta + \frac{1}{2\pi i}H_{\mathscr{B}}\right\}\mathscr{K}^{-1}, \end{aligned} \qquad (A.3)$$

$$\Delta := \frac{1}{2\pi i}\log\Lambda_{\mathscr{B}} = \mathrm{diag}\,\{\,\underbrace{\delta_1,\dots,\delta_1}_{m_1\ \text{–times}},\dots,\underbrace{\delta_\ell,\dots,\delta_\ell}_{m_\ell\text{–times}}\,\}\,, \quad \delta_j := \frac{1}{2\pi i}\log\lambda_j\,.$$

Introducing the notation

$$B_{\pm}(x) := B_{\mathscr{B}}\left(\frac{1}{2\pi i}\log(x\pm i)\right)\,, \qquad (A.4)$$

where the branch of the logarithm is fixed in the complex plane cut along the ray $\{z \in \mathbb{C} : \arg z = \gamma_0\}$, we find (cf. (A.2))

$$\begin{aligned} B_{\mathscr{B}}\left(\frac{1}{2\pi i}\log\frac{x-i}{x+i}\right) \ &= \ B_-(x)B_+^{-1}(x) \\ &= \ \begin{cases} [B_{\mathscr{B}}(1)]^{-1} + \mathscr{O}\left(|x-i|^{-1}\right) & \text{if} \quad x \to -\infty\,, \\ I_N + \mathscr{O}\left(|x+i|^{-1}\right) & \text{if} \quad x \to +\infty\,. \end{cases} \end{aligned} \qquad (A.5)$$

A.2. Hölder and Zygmund Spaces

We recall the definitions of some important spaces and expose their relevant properties to the present investigation.

For $s > 0$ the Zygmund space $\mathbb{Z}^s(\mathbb{R})$ is defined as the Banach space of functions with the finite norm

$$\begin{gathered} \|f\big|\mathbb{Z}^s(\mathbb{R})\| = \|f\big|C^m(\mathbb{R})\| + \sup_{h\neq 0}\left\{|h|^{-\nu}\|\Delta_h^2\partial^m f\big|C(\mathbb{R})\|\right\}, \\ s = m+\nu\,, \quad m\in\mathbb{N}_0\,, \quad 0<\nu\le 1\,, \end{gathered}$$

where $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$ and $\mathbb{N}$ denotes the set of all positive integers, $\Delta_h f(x) := f(x+h) - f(x)$, $\Delta_h^2 = \Delta_h\Delta_h$ and

$$\|f|C^m(\mathbb{R})\| = \sum_{k=0}^{m}\sup\{|\partial^k f(x)| : x\in\mathbb{R}\}.$$

For $s \in \mathbb{R}^+ \setminus \mathbb{N}$ the space $\mathbb{Z}^s(\mathbb{R})$ coincides with the Hölder space $C^s(\mathbb{R})$ (cf. [St, § V.4, Proposition 8]), which is endowed with the norm

$$\begin{gathered} \|f\big|C^s(\mathbb{R})\| = \|f\big|C^m(\mathbb{R})\| + \sup_{h\neq 0}\left\{|h|^{-\nu}\|\Delta_h\partial^m f\big|C(\mathbb{R})\|\right\}, \\ s = m+\nu, \quad m\in\mathbb{N}_0, \qquad 0<\nu<1. \end{gathered}$$

For $s = m + \nu$, $m \in \mathbb{N}_0$, $0 < \nu \le 1$, the space $H^s(\mathbb{R})$ of Hölder continuous functions on $\mathbb{R}$ consists of functions with the finite norm

$$\|\varphi\big|H^s(\mathbb{R})\| := \|\varphi\big|C^m(\mathbb{R})\| + \sup_{\substack{x,h\in\mathbb{R}\\ h\neq 0}} \frac{|\partial^m\varphi(x+h) - \partial^m\varphi(x)|}{\left|\dfrac{x+h}{x+h+i} - \dfrac{x}{x+i}\right|^\nu}\,. \tag{A.6}$$

This norm can also be written in the two following forms:

$$\begin{aligned}\|\varphi\big|H^s(\mathbb{R})\| &= \|\varphi\big|C^m(\mathbb{R})\| + \sup_{\substack{x,h\in\mathbb{R}\\ h\neq 0}} \frac{|\partial^m\varphi(x+h) - \partial^m\varphi(x)|}{\left|\dfrac{1}{x+h+i} - \dfrac{1}{x+i}\right|^\nu}\\ &= \|\varphi\big|C^m(\mathbb{R})\| + 2\sup_{\substack{x,h\in\mathbb{R}\\ h\neq 0}} \frac{|\partial^m\varphi(x+h) - \partial^m\varphi(x)|}{\left|\dfrac{x+h-i}{x+h+i} - \dfrac{x-i}{x+i}\right|^\nu}\,.\end{aligned} \tag{A.7}$$

Similarly $\mathscr{Z}^s(\mathbb{R})$ denotes the Zygmund space consisting of functions with the finite norm

$$\|\varphi\big|\mathscr{Z}^s(\mathbb{R})\| := \|\varphi\big|C^m(\mathbb{R})\| + \sup_{\substack{x,h\in\mathbb{R}\\ h\neq 0}} \frac{|\partial^m\varphi(x+h) + \partial^m\varphi(x-h) - 2\partial^m\varphi(x)|}{\left|\dfrac{x+h}{x+h+i} - \dfrac{x}{x+i}\right|^\nu}\,. \tag{A.8}$$

The space $H^\nu(\mathbb{R})$ differs from the above defined $C^\nu(\mathbb{R})$ since $\mathbb{R}$ is not compact; for compact curves Γ the spaces $H^\nu(\Gamma)$ and $C^\nu(\Gamma)$ are isomorphic.

For $s \in \mathbb{R}^+ \setminus \mathbb{N}$ the Zygmund space $\mathscr{Z}^s(\mathbb{R})$ coincides with the Hölder space $H^s(\mathbb{R})$ and differs for $s = 1, 2, \ldots$ ($\mathscr{Z}^s(\mathbb{R})$ contains $H^s(\mathbb{R})$ as a proper subspace; cf. [St, § V.4, Proposition 8] for details). The advantage of the Zygmund space $\mathscr{Z}^s(\mathbb{R})$ (compared with $H^s(\mathbb{R})$) is that the scale $\{\mathscr{Z}^s(\mathbb{R})\}_{s>0}$ allows interpolation (cf. [Tr]).

For a positive $\mu > 0$, $\mu = m + \nu$, $m \in \mathbb{N}$, $0 < \nu \le 1$ we consider the following Banach algebra

$$\mathscr{H}^\mu(\mathbb{R}) := \{\varphi \in C^m(\mathbb{R}) : (x+i)^k\partial_x^k\varphi \in \mathscr{Z}^\nu(\mathbb{R}), k = 0, 1, \ldots, m\}\,,$$

endowed with the norm

$$\|\varphi\big|\mathscr{H}^\mu(\mathbb{R})\| := \sum_{k=0}^m \|(x+i)^k\partial_x^k\varphi\big|\mathscr{Z}^\nu(\mathbb{R})\|\,.$$

If $\varphi \in \mathscr{H}^\mu(\mathbb{R})$ by sending in (A.7) $x \to 0$ we get

$$\partial_x^k[\varphi(h) - \varphi(\infty)] = \mathscr{O}\left(\langle h\rangle^{-\nu-k}\right)\,, \quad k = 0, 1, \ldots, m\,. \tag{A.9}$$

Obviously,

$$g(-\infty) = \lim_{x\to-\infty} g(x) = \lim_{x\to+\infty} g(x) = g(+\infty) \tag{A.10}$$

for all functions $g \in \mathscr{Z}^\mu(\mathbb{R})$ and $g \in \mathscr{H}^\mu(\mathbb{R})$ (cf. the definition of norms (A.6) and (A.8)). Therefore the Banach algebra $\mathscr{Z}^\mu(\overline{\mathbb{R}})$ of functions

$$\varphi(x) = \omega(x)\varphi_-(x) + [1 - \omega(x)]\varphi_+(x)\,, \qquad \varphi_\pm \in \mathscr{Z}^\mu(\mathbb{R})\,, \tag{A.11}$$

where

$$\omega \in C^\infty(\mathbb{R}), \qquad \omega(x) = 1 \quad for \quad x < -1, \qquad \omega(x) = 0 \quad for \quad x > 1,$$

differs from the space $\mathscr{Z}^\mu(\mathbb{R})$ since the function $\varphi(x)$ in (A.11) has, in general, different limits:

$$\varphi(-\infty) = \varphi_-(-\infty), \qquad \varphi(+\infty) = \varphi_+(+\infty). \tag{A.12}$$

The Banach algebra $\mathscr{H}^\mu(\overline{\mathbb{R}})$ is defined similarly.

For $0 < \nu \leq 1$ the spaces $\mathscr{Z}^\nu(\mathbb{R})$ and $\mathscr{Z}^\nu(\Gamma_0)$, where $\Gamma_0 = \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle, are isomorphic:

$$\varpi_* \ : \ \mathscr{Z}^\nu(\mathbb{R}) \longrightarrow \mathscr{Z}^\nu(\Gamma_0), \quad \varpi_*\varphi(z) := \varphi\left(i\frac{1+z}{1-z}\right), \quad z \in \Gamma_0. \tag{A.13}$$

The inverse isomorphism reads

$$\varpi_*^{-1}\psi(x) := \psi\left(\frac{x-i}{x+i}\right), \quad x \in \mathbb{R}.$$

In fact,

$$\begin{aligned}\|\varpi_*\varphi\big|\mathscr{Z}^\nu(\Gamma_0)\| &= \sup_{z\in\Gamma_0}\left|\varphi\left(i\frac{1+z}{1-z}\right)\right| \\ &+ \sup_{\substack{z,z_{\pm h}\in\Gamma_0 \\ |z_h-z|=|z_{-h}-z|\neq 0}} \frac{\left|\varphi\left(i\frac{1+z_h}{1-z_h}\right)+\varphi\left(i\frac{1+z_{-h}}{1-z_{-h}}\right)-2\varphi\left(i\frac{1+z}{1-z}\right)\right|}{|z_h-z|^\nu} \\ &= \sup_{x\in\mathbb{R}}|\varphi(x)| + \sup_{\substack{x,h\in\mathbb{R}\\ h\neq 0}} \frac{|\varphi(x+h)+\varpi(x-h)-2\varphi(x)|}{\left|\frac{x+h-i}{x+h+i}-\frac{x-i}{x+i}\right|^\nu}\end{aligned}$$

and, due to (A.7),

$$\|\varphi\big|\mathscr{Z}^\nu(\mathbb{R})\| \leq \|\varpi_*\varphi\big|\mathscr{Z}^\nu(\Gamma_0)\| \leq 2\|\varphi\big|\mathscr{Z}^\nu(\mathbb{R})\|.$$

The next Proposition states a certain inverse estimates to (A.9).

Proposition A.1. *Let* $0 < \nu \leq 1$, $m \in \mathbb{N}_0$.

(i) *If* $\varphi \in C^m(\mathbb{R})$ *and*

$$C_{k,\nu} := \sup_x \big||x+i|^{k+\nu}\partial_x^k[\varphi(x)-\varphi(\infty)]\big| < \infty \quad for \quad k = 0,1,...,m,$$

then $\varphi \in \mathscr{H}^{m-1+\nu}(\mathbb{R})$ *and* $\|\varphi\big|\mathscr{H}^{m-1+\nu}(\mathbb{R})\| \leq M\sum\limits_{k=0}^{m} C_{k,\nu}$, *where* M *=const is independent of* φ.

(ii) *If* $\varphi \in \widetilde{\mathscr{H}}^{m+\nu}(\mathbb{R})$ *and*

$$\partial_x^k b(x) = \mathscr{O}\left(\langle x\rangle^{-k}\right) \quad for \quad k = 0,1,...m, \tag{A.14}$$

then $b\varphi \in \widetilde{\mathscr{H}}^{m+\nu}(\mathbb{R})$.

(iii) *If* $\varphi \in \widetilde{\mathscr{Z}^\nu}(\mathbb{R})$, $0 < \nu \leq 1$, *and if* (A.14) *holds, then* $b\varphi \in \widetilde{\mathscr{Z}^\nu}(\mathbb{R})$.
(iv) *If* $\varphi \in \widetilde{\mathscr{Z}^\nu}(\mathbb{R})$, $0 < \theta < \nu \leq 1$, *then* $(x+i)^\theta \varphi \in \widetilde{\mathscr{Z}^{\nu-\theta}}(\mathbb{R})$.

Proof. For the proof of Proposition A.1.i and Proposition A.1.ii we refer to [ChDu, § 1.6]. Proposition A.1.iii and Proposition A.1.iv are proved by analogy to Proposition A.1.ii, based on similar assertions proved in [Mu, Chapt.1, § 6] for a smooth curve. ☐

Remark A.2. As an example of the function $b(x)$ in (A.15) we can take $(x+i)^{i\mu}$, $\mu \in \mathbb{R}$.

Corollary A.3. *If* $0 < \mu_1 \leq \mu_2$, *the embedding* $\mathscr{H}^{\mu_2}(\mathbb{R}) \subset \mathscr{H}^{\mu_1}(\mathbb{R})$ *is continuous.*

Proof. The claim follows from the foregoing Proposition A.1 and from the asymptotic property (A.9). ☐

Rational functions

$$r_\ell(x) = \sum_{|k|\leq\ell} c_k \left(\frac{x-i}{x+i}\right)^k , \quad x \in \mathbb{R}\,, \quad c_k \in \mathbb{C} \tag{A.15}$$

belong to all $\mathscr{H}^\mu(\mathbb{R})$ (see Proposition A.1). Let $\widetilde{\mathscr{H}^\mu}(\mathbb{R})$ denote the sub-algebra of $\mathscr{H}^\mu(\mathbb{R})$ obtained by closing the algebra of rational functions (A.15). The algebra $\widetilde{\mathscr{H}^\mu}(\mathbb{R})$ is rationally dense by the definition in [BuGo] (see also [ClGo]).

In [Ta, § 1.3.4] the sub-algebra $\widetilde{\mathscr{H}^\mu}(\mathbb{R})$ is characterized for $0 < \mu < 1$ (the same holds for all non–integer $\mu \in \mathbb{R}^+ \setminus \mathbb{N}_0$) as follows: $\varphi \in \widetilde{\mathscr{H}^\mu}(\mathbb{R})$ iff

$$\lim_{\varepsilon\to 0} \sup_{\substack{|x'-x|<\varepsilon \\ x'\neq x}} \frac{|\varphi(x') - \varphi(x)|}{\left|\dfrac{x'}{x'+i} - \dfrac{x}{x+i}\right|^\nu} = 0$$

uniformly for all $x \in \mathbb{R} \cup \{\infty\}$.

Proposition A.4. *If* $0 < \mu = m + \nu < \mu' = m' + \nu'$, $m, m' \in \mathbb{N}_0$, $0 < \nu, \nu' < 1$, *then the embedding* $\mathscr{H}^{\mu'}(\mathbb{R}) \subset \widetilde{\mathscr{H}^\mu}(\mathbb{R})$ *is continuous and dense.*

If $\varphi \in \widetilde{\mathscr{H}^\mu}(\mathbb{R})$ *and*

$$\partial_x^k b(x) = \mathscr{O}\left(|x+i|^{-k}\right) \quad \textit{for} \quad k = 0, 1, ...m\,,$$

then $b\varphi \in \widetilde{\mathscr{H}^\mu}(\mathbb{R})$.

Proof. For the proof we refer to [ChDu, § 1.6]. ☐

Let

$$\mathscr{Z}_0^\nu(\mathbb{R}) := \{\varphi \in \mathscr{Z}^\nu(\mathbb{R}) \ : \ \varphi(\infty) = 0\}\ .$$

Theorem A.5. *Let* $\mu \in \mathbb{R}^+$. *Then the Hilbert transform*

$$H_\mathbb{R}\varphi(x) := \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\varphi(\tau)\, d\tau}{\tau - x} \tag{A.16}$$

is bounded in the spaces $\mathscr{Z}^\nu(\mathbb{R})$, *in* $\mathscr{Z}_0^\nu(\mathbb{R})$, *in* $\mathscr{H}^\nu(\mathbb{R})$ *and in* $\widetilde{\mathscr{H}^\nu}(\mathbb{R})$ *for all* $\nu \geq 0$.

Proof. Let us prove that $H_{\mathbb{R}}$ is a bounded operator in the space $\mathscr{Z}^{\nu}_0(\mathbb{R})$. Then, due to the relations

$$\mathscr{Z}^{\nu}(\mathbb{R}) = \{\mathrm{const}\}\dot{+}\mathscr{Z}^{\nu}_0(\mathbb{R})\,, \quad H_{\mathbb{R}}c = 0 \ \text{ for } \ c = \mathrm{const}\,, \tag{A.17}$$

$H_{\mathbb{R}}$ is bounded in $\mathscr{Z}^{\nu}_0(\mathbb{R})$ as well. From (A.17) follows, in particular, that

$$H_{\mathbb{R}}\varphi = H_{\mathbb{R}}\varphi_0\,, \qquad \varphi_0(x) := \varphi(x) - \varphi(\infty) \tag{A.18}$$

for arbitrary $\varphi \in \mathscr{Z}^{\nu}(\mathbb{R})$. Then, integrating by parts,

$$\partial H_{\mathbb{R}}\varphi = \partial H_{\mathbb{R}}\varphi_0 = H_{\mathbb{R}}\partial\varphi_0 = H_{\mathbb{R}}\partial\varphi \tag{A.19}$$

which means that the Hilbert transform commutes with the derivative $\partial := d/dt$. Therefore, due to the property (A.19), it suffices to prove the theorem for $0 < \mu \leq 1$: due to (A.19) it is easily extensible to all $\mu > 0$.

Thus, we assume $0 < \mu \leq 1$.

The Cauchy singular integral operator

$$S_{\Gamma_0}\psi(z) := \frac{1}{\pi i}\int\limits_{\Gamma_0} \frac{\psi(\zeta)d\zeta}{\zeta - z}$$

is bounded in $\mathscr{Z}^{\nu}(\Gamma_0)$ for all $\nu > 0$. For $0 < \nu < 1$ this is known as *Privalov's Theorem* (see [GoKr, MiPr, Mu]), for $m < \nu < m+1$ (with $m = 1, 2, \ldots$) it follows by a property similar to (A.19), and for $\nu = m$ it can be derived by interpolation (cf. [DuSp, CaDuSp, St]).

The operator $\varpi_*^{-1}S_{\Gamma_0}\varpi_*$, transformed by the isomorphism (A.13), acquires the form

$$\begin{aligned}
\varpi_*^{-1}S_{\Gamma_0}\varpi_*\varphi(x) &= \frac{1}{\pi i}\int\limits_{-\infty}^{\infty} \frac{\varphi(\tau)}{\dfrac{\tau - i}{\tau + i} - \dfrac{x - i}{x + i}}\,\frac{2i d\tau}{(\tau + i)^2} = \frac{1}{\pi i}\int\limits_{-\infty}^{\infty} \frac{x + i}{\tau + i}\,\frac{\varphi(\tau)d\tau}{\tau - x} \\
&= H_{\mathbb{R}}\varphi(x) - K_1\varphi\,, \qquad\qquad K_1\varphi := \frac{1}{\pi i}\int\limits_{-\infty}^{\infty} \frac{\varphi(\tau)d\tau}{\tau + i}\,.
\end{aligned}$$

Since the one–dimensional operator K_1 is bounded in $\mathscr{Z}^{\mu}(\mathbb{R}) \to \mathbb{C} \subset \mathscr{Z}^{\mu}(\mathbb{R})$, the operator $H_{\mathbb{R}}$ is bounded in $\mathscr{Z}^{\mu}(\mathbb{R})$ for all $0 < \mu \leq 1$.

Next we prove that $H_{\mathbb{R}}$ is bounded in $\mathscr{H}^{\mu}(\mathbb{R})$. For this we apply the integration by parts

$$(x + i)^k\partial_x^k H_{\mathbb{R}}\varphi = H_{\mathbb{R}}(x + i)^k\partial_x^k\varphi\,.$$

Applying the proved part of the theorem we proceed as follows:

$$\begin{aligned}
\|H_{\mathbb{R}}\varphi\big|\mathscr{H}^{\mu}(\mathbb{R})\| &= \sum_{k=1}^{m}\|(x + i)^k\partial_x^k H_{\mathbb{R}}\varphi\big|\mathscr{Z}^{\nu}(\mathbb{R})\| = \sum_{k=1}^{m}\|H_{\mathbb{R}}(x + i)^k\partial_x^k\varphi\big|\mathscr{Z}^{\nu}(\mathbb{R})\| \\
&\leq \|H_{\mathbb{R}}\|\sum_{k=1}^{m}\|(x + i)^k\partial_x^k\varphi\big|\mathscr{Z}^{\nu}(\mathbb{R})\| = \|H_{\mathbb{R}}\|\|\varphi\big|\mathscr{H}^{\mu}(\mathbb{R})\|\,.
\end{aligned}$$

$H_{\mathbb{R}}$ is bounded in $\widetilde{\mathscr{H}^{\mu}}(\mathbb{R})$ because it is bounded in $\mathscr{H}^{\mu'}(\mathbb{R})$ for all $0<\mu<\mu'$ (see Proposition A.4). ☐

Proposition A.6. *Let $\mu>0$ and $\mathscr{A}\in\mathscr{Z}^{\mu}(\mathbb{R})$ (or $\mathscr{A}\in\widetilde{\mathscr{H}^{\mu}}(\mathbb{R})$) be an elliptic matrix-valued function. Then $\mathscr{A}$ admits the classical factorization*

$$\begin{gathered}\mathscr{A}(x)=[\mathscr{A}_-(x)]^{-1}\left(\frac{x-i}{x+i}\right)^{\varkappa}\mathscr{A}_+(x)\,,\\ \varkappa=(\varkappa_1,\dots,\varkappa_N)\in\mathbb{Z}^N,\quad \mathbb{Z}=\{0,\pm1,\dots\}\end{gathered}\tag{A.20}$$

with factors $[\mathscr{A}_-(x)]^{\pm1}$, $[\mathscr{A}_+(x)]^{\pm1}$ which belong to $\mathscr{Z}^{\mu}(\mathbb{R})$ (or to $\widetilde{\mathscr{H}^{\mu}}(\mathbb{R})$, respectively) and have uniformly bounded analytic continuations into the half–planes $\Im m\ x<0$ and $\Im m\ x>0$, respectively.

Proof. For the proof we refer to [BuGo, ClGo]: for the space $\mathscr{Z}^{\mu}(\mathbb{R})$ the proof in [BuGo, ClGo] is direct, while for the space $\widetilde{\mathscr{H}^{\mu}}(\mathbb{R})$ it follows from the general theorem on factorization in a rationally dense and decomposable Banach algebra (cf. Proposition A.4 and Theorem A.5). ☐

Acknowledgment

The work was supported in part by "FCT–Portuguese Science Foundation", through the *Applied Mathematics Center* (Instituto Superior Técnico, U.T.L.) and Research Unit *Mathematics and Applications* (University of Aveiro).

References

[Ar] V. Arnold, *Ordinary Differential Equations*. Springer–Verlag, Heidelberg 1992 (Russian original: 3-rd edition, Nauka, Moscow 1984).

[BöSi] A. Böttcher, B. Silbermann, *Analysis of Toeplitz Operators*. Springer–Verlag, Heidelberg 1990.

[BuGo] M. Budjanu, I. Gohberg, General theorems on the factorization of matrix functions. II: Certain tests and their consequences. *Mat. Issled.* 3 (1968), 3–18.

[CaDuSp] L.P. Castro, R. Duduchava, F.-O. Speck, Singular integral equations on piecewise smooth curves in spaces of smooth functions. *Operator Theory: Advances and Applications* **135**, 107–144. Birkhäuser–Verlag, Basel 2002.

[CaSp] L.P. Castro, F.-O. Speck, Inversion of matrix convolution type operators with symmetry. *Port. Math. (N.S.)* **62** (2005), 193–216.

[CaSpTe1] L.P. Castro, F.-O. Speck, F.S. Teixeira, On a class of wedge diffraction problems posted by Erhard Meister. In: *Operator Theoretical Methods and Applications to Mathematical Physics* (Eds. I. Gohberg et al.). *Operator Theory: Advances and Applications* **147**, 211–238. Birkhäuser–Verlag, Basel 2004.

[CaSpTe2] L.P. Castro, F.-O. Speck, F.S. Teixeira, A direct approach to convolution type operators with symmetry. *Math. Nach.* **269-270** (2004), 73–85.

[CaSpTe3] L.P. Castro, F.-O. Speck, F.S. Teixeira, Mixed boundary value problems for the Helmholtz equation in a quadrant. *Integr. Equ. Oper. Theory*, 44 p., to appear.

[ChDu] O. Chkadua, R. Duduchava, Pseudodifferential equations on manifolds with boundary: Fredholm property and asymptotic. *Math. Nachr.* **222** (2001), 79–139.

[ClGo] K. Clancey, I. Gohberg, *Factorization of Matrix Functions and Singular Integral Operators*. *Operator Theory: Advances and Applications* **3**. Birkhäuser–Verlag, Basel 1981.

[DuSp] R. Duduchava, F.–O. Speck, Pseudo–differential operators on compact manifolds with Lipschitz boundary, *Math. Nachr.* **160** (1993), 149–191.

[DuSäWe] R. Duduchava, A.M. Sändig, W. Wendland, Interface cracks in anisotropic composites, *Math. Meth. Appl. Sciences* **22** (1999), 1413–1446.

[Eh] T. Ehrhardt, Invertibility theory for Toeplitz plus Hankel operators and singular integral operators with flip. *J. Funct. Anal.* **208** (2004), 64–106.

[Ga] F. Gantmacher, *Matrix Theory*. Nauka, Moscow, 1967.

[GoKr] I. Gohberg, N. Krupnik, *Introduction to the theory of one–dimensional singular integral operators*. Birkhäuser–Verlag, Basel 1992.

[GoKaSp] I. Gohberg, M.A. Kaashoek, I.M. Spitkovsky, *An overview of matrix factorization theory and operator applications*. In: *Factorization and Integrable Systems* (Eds. I. Gohberg et al.). Lecture notes of the summer school, Faro, Portugal, September 2000. *Operator Theory: Advances and Applications* **141**, Birkhäuser–Verlag, Basel 2003, p. 1–102.

[Kr] N.Ya. Krupnik, *Banach Algebras with Symbol and Singular Integral Operators*. Birkhäuser–Verlag, Basel 1987.

[La] P. Lancaster, *Theory of Matrices*. Academic Press, New York 1969.

[LiSp] G.S. Litvinchuk, I.M. Spitkovsky, *Factorization of Measurable Matrix Functions*. Birkhäuser–Verlag, Basel 1987.

[MiPr] S. Mikhlin, S. Prössdorf, *Singular Integral Operators*. Springer–Verlag, Heidelberg 1986.

[Mu] N. Muskhelishvili, *Singular Integral Equations*, Nordhoff, Groningen 1953. Last Russian edition: Nauka, Moscow 1968; Last English edition: Dover Publications, Inc., New York 1992.

[Sh] E. Shamir, Elliptic systems of singular integral equations. I: The half-space case. *Trans. Amer. Math. Soc.* **127** (1967), 107–124.

[Si] I.B. Simonenko, Some general questions in the theory of the Riemann boundary problem (Russian). Izv. Akad. Nauk SSSR Ser. Mat. **32** (1968), 1138–1146. English translation in Math. USSR, Izv. **2** (1968), 1091–1099.

[St] E. Stein, *Singular Integrals and Differentiability Properties of Functions*. Princeton Univ. Press, Princeton 1970.

[Ta] N. Tarkhanov, *The Cauchy Problem for Solutions of Elliptic Equations*. Akademie-Verlag, Berlin 1995.

[Tr] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*. North-Holland, Amsterdam 1978.

L.P. Castro
Department of Mathematics
University of Aveiro
3810-193 Aveiro
Portugal
e-mail: `lcastro@mat.ua.pt`

R. Duduchava
A. Razmadze Mathematical Institute
Academy of Sciences of Georgia
1, M.Alexidze str.
Tbilisi 93
Georgia
e-mail: `dudu@rmi.acnet.ge`

F.-O. Speck
Department of Mathematics
Instituto Superior Técnico, U.T.L.
Avenida Rovisco Pais
1049–001 Lisboa
Portugal
e-mail: `fspeck@math.ist.utl.pt`

Recent Preprints

2005

1/05 Natasha Samko. *On non-equilibrated almost monotonic functions of the Zygmund-Bari-Stechkin class*. 18pp.

2/05 Natasha Samko. *Singular integral operators in weighted spaces of continuous functions with non-equilibrated continuity modulus*. 26pp.

3/05 Pedro Resende. *Étale groupoids and their quantales*. 62pp.

4/05 M. Casquilho, M. Constantino, M. C. Morais. *Sobre a amostragem de aceitação para a variável gaussiana inversa*. 14pp.

5/05 M. C. Morais, A. Pacheco. *Assessing the impact of head starts in the performance of one-sided Markov-type control schemes*. 23pp.

6/05 L. P. Castro, F.–O., Speck, F. S. Teixeira. *Mixed boundary value problems for the Helmholtz equation in a quadrant*. 42pp.

7/05 M. A. Bastos, P. A. Lopes, A. M. Santos. *Periodic oblique derivative and Neumann boundary-value problems: the two straight line approach*. 22pp.

8/05 Natasha Samko. *On compactness of integral operators with a generalized weak singularity in weighted spaces of continuous functions with a given continuity modulus*. 25pp.

9/05 M. C. Morais, A. Pacheco, W. Schmid. *On the stochastic behaviour of the run length of EWMA control schemes for mean of correlated output in the presence of shifts in σ.* 20pp.

10/05 R. Coutinho, B. Fernandez, R. Lima, A. Meyroneinc. *Discrete time piecewise affine models of genetic regulatory networks*. 37pp.

11/05 Yu. I. Karlovich and Luís Pessoa. *C*-Algebras of Bergman Type Operators with Piecewise Continuous Coefficients*. 53pp.

12/05 Jean-François Babadjian and Margarida Baía. *Multiscale nonconvex relaxation and application to thin films*. 49pp.

The Mathematics Department of Instituto Superior Técnico offers a preprint series open to its faculty, PhD students and visitors who wish to make available a preview of their work before formal publication. Starting with issue 11/05 the preprints are only available in electronic (PDF) format from

`http://preprint.math.ist.utl.pt/?serie=dmist`